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FLOWS RESULTING FROM THE INCIDENCE OF A DISCONTINUOUS WAVE ON A BOTTOM STEP

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The solvability of the problem of the flows resulting from the incidence of a discontinuous wave on a bottom step is studied using a single-layer shallow water model. Solutions in which the total energy of the flow is conserved at the step and those in which it is lost at the step are considered. Regions of double and triple hystereses in the obtained self-similar solutions are found. An analogy is drawn with the problem of single-layer flow over a bottom obstacle.

Key words: *shallow water, discontinuous wave, bottom step.*

1. Formulation of the Problem. In the case of a rectangular channel of constant width and variable depths and ignoring friction, the single-layer shallow water differential equations [1–4] are written as

$$h_t + q_x = 0; \tag{1.1}$$

$$q_t + (qv)_x + ghz_x = 0, \tag{1.2}$$

where $h(x,t)$, $q(x,t)$, $v = q/h$, and $z = b + h$ are the fluid depth, flow rate, velocity, and level, respectively, $b(x)$ is the coordinate of the channel bottom, and g is the gravity acceleration. Equation (1.1) represents the mass conservation law, and Eq. (1.2) the conservation law for the total momentum. These conservation laws imply the Hugoniot conditions [1, 4]:

$$D[h] = [q]; \tag{1.3}$$

$$D[q] = [qv + gh^2/2], \tag{1.4}$$

which link the flow parameters of a discontinuous wave propagating at a speed D over an even bottom. In formulas (1.3) and (1.4), $[f]$ denotes the jump of the function f at the discontinuous wave front.

For system (1.1), (1.2), we consider the problem of initial discontinuity decay

$$z(x,0) = \begin{cases} z_1, & x < 0, \\ z_0, & x > 0, \end{cases} \qquad z_1 > z_0, \qquad v(x,0) = \begin{cases} v_1, & x < 0, \\ 0, & x > 0, \end{cases} \qquad v_1 > 0 \tag{1.5}$$

above a bottom level jump

$$b(x) = \begin{cases} 0, & x < 0, \\ \delta, & x > 0, \end{cases} \qquad z_0 > \delta > 0. \tag{1.6}$$

Here the flow parameters $z_1 = h_1, v_1$ to the left of the discontinuity (1.6) satisfy the condition

$$v_1 = v_s(z_1, z_0), \tag{1.7}$$

in which

$$v = u_s(h) = v_s(h, z_0) = (h - z_0)\sqrt{g(h + z_0)/(2hz_0)}, \quad h > z_0 \tag{1.8}$$

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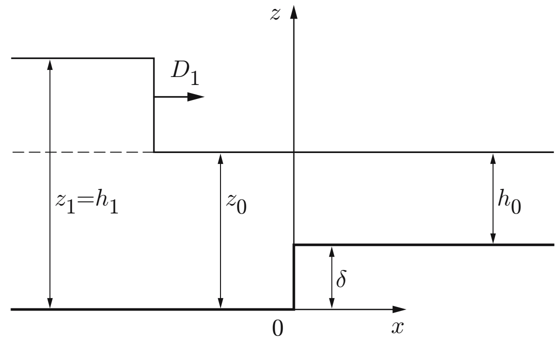

Fig. 1. Profile of the initial discontinuous wave incident on the bottom step.

is the equation of the shock s-adiabat [4]. Equation (1.8), which is obtained from the Hugoniot conditions (1.3) and (1.4), links the flow parameters $h_0 = z_0$ and $v_0 = 0$ ahead of the discontinuous wave front with their possible values (h and v) behind its front. Since $z_1 > z_0$, $v_1 > 0$, and $q(0,t) > 0$ at $t > 0$ and, in view of the nomenclature adopted in [5], the discontinuity (1.6) represents the bottom step on which water flows. Thus, in view of (1.7) and (1.8), the self-similar solutions of the problem (1.1)–(1.6) describe the flows resulting from the incidence of a discontinuous wave on a bottom step (Fig. 1). The speed of propagation of this initial discontinuous wave is calculated by the formula

$$D_1 = D(z_1, z_0) = \sqrt{gz_1(z_1 + z_0)/(2z_0)}. \qquad (1.9)$$

For $x < 0$, the solution of the problem (1.5)–(1.8) will be called the flow to the left of the step, and for $x > 0$, it will be called the flow to the right of the step; for $x = 0 - 0$, the value of the exact solution at the discontinuity (1.6) will be called the flow ahead of the step, and for $x = 0 + 0$, it will be called the flow at the step.

The problem (1.5)–(1.8) is a particular case of the general problem of the arbitrary discontinuity decay above a bottom level jump, which was studied in [6], where qualitatively different examples of its solution were constructed under the assumption that the total energy of the flow is conserved at the discontinuity (1.6). However, in [6] the uniqueness of these solutions was not studied and the regions of their existence were not found. The unique solvability of the problem (1.5), (1.6) for $v_1 = 0$ (the problem of dam break above a bottom step [7]) was studied in [8], and a comparison with results of laboratory experiments is given in [9]. Self-similar solutions of the problem (1.5), (1.6) for $v_1 = 0$ and $\delta < 0$, where it becomes the problem of dam break above a bottom step, are constructed in [10, 11], and a comparison of these solutions with experiment is performed in [12].

In the present paper, which is a continuation of [8, 11], we study the solvability of the generalized problem (1.5)–(1.8) of the flows resulting form the incidence of a discontinuous wave on a bottom step. Because the shallow water equations (1.1) and (1.2) are a simple example of the strongly nonlinear hyperbolic system of conservation laws [13] that is equivalent to the system of equations of isentropic gas dynamics [14] with an isentropic exponent $\gamma = 2$, the solution for these equations of the generalized problem of discontinuity decay (1.5)–(1.8) is sought, following [15], in the form of a combination of simple waves, a stationary jump located at the coordinate origin above the bottom step, and constant-flow regions connecting them. This is done using the generalized method of adiabats, which was first time used in [16] to solve the problem of the decay of a gas-dynamic discontinuity in a channel with a cross section jump. This method gives four qualitatively different types of steady-state self-similar solutions of the problem (1.5)–(1.8): in three of them the total energy of the flow is conserved at the bottom step, and in one it is lost at the bottom step. The regions of existence of these solutions are plotted on the plane of the dimensionless determining parameters δ and z_1 obtained for $g = z_0 = 1$. Subregions of double and triple hystereses, i.e., subregions in which two or three different self-similar solutions occur simultaneously, are found. The solutions constructed are compared with the solutions of the problem of single-layer flow over a long obstacle at the bottom [3, 17–19].

2. Self-Similar Solutions with a Reflected Discontinuous Wave. To construct the self-similar solutions of the discontinuity decay problem formulated in Sec. 1, it is necessary to specify relations for the flow parameters at the discontinuity arising above the bottom step (1.6). Following [8, 11], for such a discontinuity we

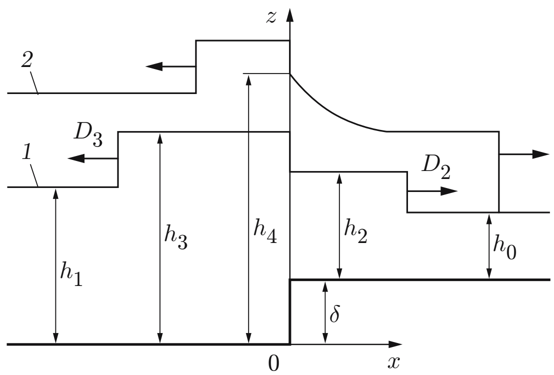

Fig. 2

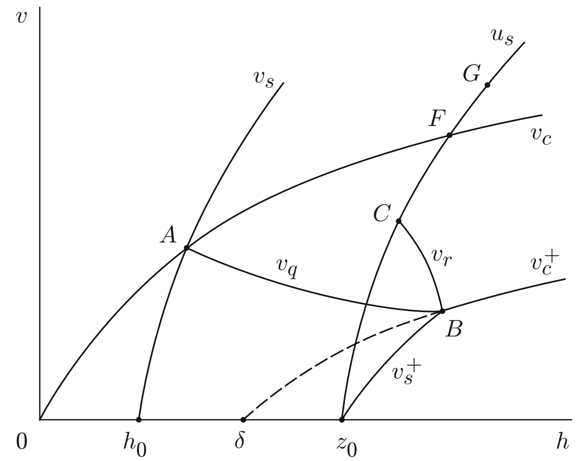

Fig. 3

Fig. 2. Wave profiles arising after the passage of the initial discontinuous wave above the bottom step for the case of subcritical flow ahead of the step: 1) flow described by a solution of type A; 2) flow described by a solution of type B.

Fig. 3. Diagram of adiabats for constructing solutions of type A and B.

first assume the satisfaction of the mass conservation law (1.1) and the conservation law for the local momentum

$$v_t + (v^2/2 + gz)_x = 0, \tag{2.1}$$

and hence, as shown in [20], the conservation law for the total energy

$$e_t + (q(v^2/2 + gz))_x = gbq_x, \tag{2.2}$$

where $e = (qv+gh^2)/2$ is the total energy of the incident flow. Equations (2.1) and (2.2) are differential consequences of system (1.1), (1.2) for its smooth solutions. The Hugoniot conditions for the conservation laws (1.1) and (2.1) at a standing jump with a propagation speed of $D = 0$ are written as

$$[q] = 0, \qquad [v^2/2 + gz] = 0, \tag{2.3}$$

i.e., at such a jump, the flow rate and the Bernoulli constant are continuous.

The depth and flow velocity at the step will be denoted as h and v and those ahead of the step by H and V, respectively. Then, relations (2.3) are written as

$$J(H, q) = J(h, q) + \delta, \qquad q = hv = HV, \tag{2.4}$$

where $J(y, q) = q^2/(2gy^2) + y$. As shown in [20], relations (2.4) specify two mappings F_+ and F_-. The first mappings F_+ transforms each flow at the step (with parameters h and v) to subcritical flow (with H_+ and V_+) and the second mapping F_- transforms to supercritical flow ahead of the step (with parameters H_- and V_-). In the region of subcritical and critical flows ($v \le \sqrt{gh}$), the steady-state discontinuity above the step is specified by the mapping F_+, and in the region of supercritical flows ($v > \sqrt{gh}$), it is specified by the mapping F_-.

After the initial discontinuous wave has passed over the bottom step, a new discontinuous wave propagates to the right of the step (Fig. 2); the flow parameters behind the front of this wave (h_2 and v_2) lie on the shock s-adiabat

$$v = v_s(h) = v_s(h, h_0) = (h - h_0)\sqrt{g(h + h_0)/(2hh_0)}, \qquad h > h_0. \tag{2.5}$$

In Fig. 3, the curves u_s, v_s, and v_c show the shock s-adiabats (1.8) and (2.5) and the set of critical flows

$$v = v_c(h) = \sqrt{gh}, \tag{2.6}$$

and the curves $v_c^+ = F_+[v_c]$ and $v_s^+ = F_+[v_s]$ show the images obtained of the set (2.6) and the part of the shock s-adiabat (2.5) lying in the region of subcritical flows (below the point A in Fig. 3) under the mapping F_+.

Let us introduce the auxiliary function

$$v = v^+(h) = \begin{cases} v_s^+(h), & z_0 < h \le h_B, \\ v_c^+(h), & h \ge h_B, \end{cases} \tag{2.7}$$

where $B = (h_B, v_B)$ is the point of intersection of the plots of the functions v_s^+ and v_c^+. Since, as shown in [8], the functions v_s^+ and v_c^+ are strictly monotonically increasing, the function v^+ is also strictly monotonically increasing. The solution of the problem (1.5)–(1.8) to the left of the step depends largely on the relative position of the shock s-adiabat (1.8) and curve (2.7) in Fig. 3. The following theorem holds.

Theorem 1. *The functions $u_s(h)$ and $v^+(h)$ satisfy the inequality*

$$u_s(h) > v^+(h) \quad \forall h > z_0. \tag{2.8}$$

This theorem is proved in Sec. 6. Theorem 1 implies that for $h > z_0$, the plot of the adiabat u_s in Fig. 3 lies above the plot of the function v^+.

On the adiabat u_s we fix the point $C = (h_C, v_C)$ of origin of the shock r-adiabat

$$v = v_r(h, h_C, v_C) = v_C - (h - h_C)\sqrt{g(h + h_C)/(2hh_C)}, \qquad h > h_C,$$

which passes through the point B. The coordinates of the point C are found from the system of equations $v_C = u_s(h_C)$ and $v_B = v_r(h_B, h_C, v_C)$. The following theorem holds.

Theorem 2. *The flow (h_C, v_C) is subcritical, i.e., $v_C < \sqrt{gh_C}$.*

This theorem is proved in Sec. 7. Theorem 2 implies that the point $C = (h_C, v_C)$ lies on the part of the adiabat u_s that lies below the critical flow curve v_c in Fig. 3.

We assume that the flow parameters (h_1 and v_1) behind the front of the initial discontinuous wave incident on the step are not above the point C on the adiabat u_s, by virtue of which the depth h_1 satisfies the inequalities $z_0 < h_1 \le h_C$. Then, the solution of the discontinuity decay problem (1.5)–(1.8) yields a reflected discontinuous wave propagating against the background (h_1 and v_1) with the flow parameters (h_3 and v_3) behind the front of this wave uniquely determined from the equations $v_3 = v_r(h_3, h_1, v_1) = v_s^+(h_3)$ as the coordinates of the point of intersection of the strictly monotonically decreasing shock r-adiabat $v = v_r(h, h_1, v_1)$ with origin at the point (h_1, v_1) on the adiabat u_s with the plot of the strictly monotonically increasing function v_s^+. In this case, the subcritical or critical constant flow behind the front of the discontinuous wave propagating behind the step starts directly from the step and its parameters h_2 and v_2 are calculated as the coordinates of the point of intersection of the adiabat v_s with the hyperbola

$$v = v_q(h, h_3, v_3) = q/h \qquad (q = h_3 v_3) \tag{2.9}$$

with origin at the point (h_3, v_3).

The speed of the discontinuous wave propagating behind the step is determined with allowance for (1.9) from the formula

$$D_2 = D(h_2, h_0) = \sqrt{gh_2(h_2 + h_0)/(2h_0)}, \tag{2.10}$$

and the speed of the reflected discontinuous wave is found from the formula

$$D_3 = v_1 - D(h_3, h_1) = v_1 - \sqrt{gh_3(h_3 + h_1)/(2h_1)}. \tag{2.11}$$

The resulting solution, whose profile is shown in Fig. 2 by curve 1, will be called flow of type A.

We now assume that $h_1 > h_C$, i.e., the point (h_1, v_1) lies above the point C on the adiabat u_s (see Fig. 3). Then, the adiabat $v = v_r(h, h_1, v_1)$ intersects the plot of the function (2.7) in the line v_c^+, by virtue of which the flow parameters (h_3, v_3) behind the front of the reflected discontinuous wave are uniquely determined from the equations $v_3 = v_r(h_3, h_1, v_1) = v_c^+(h_3)$. In this case, the flow at the step is critical and its parameters are $h_4 = \sqrt[3]{q^2/g}$ and $v_4 = \sqrt[3]{gq}$, where $q = h_3 v_3$, are calculated as the coordinates of the point of intersection of hyperbola (2.9) with the critical-flow curve (2.6). The flow to the right of the step is supercritical and does not influence the flow parameters at the step and to the left of it. To construct the solution describing this flow, it is necessary to solve the classical problem of discontinuity decay above a horizontal bottom for the system of shallow water equations [1, 4] with the following initial data:

$$h(x,0) = \begin{cases} h_4, & x \le 0, \\ h_0, & x > 0, \end{cases} \qquad v(x,0) = \begin{cases} v_4, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{2.12}$$

The solution of the problem (2.12) yields a discontinuous s-wave and a centered depression r-wave, which are connected by a region of constant flow, whose parameters h_2 and v_2 are determined as the coordinates of the point of intersection of the shock s-adiabat (2.5) with the wave r-adiabat

$$v = v_r(h, h_4, v_4) = v_4 + 2\sqrt{g}\,(\sqrt{h_4} - \sqrt{h}\,), \qquad h < h_4. \tag{2.13}$$

Adiabat (2.13) issues from the point (h_4, v_4), which lies on the critical-flow curve v_c to the right of the point A (see Fig. 3) at which this curve is intersected with the shock adiabat v_s. The flow parameters in the centered depression r-wave are calculated by the formulas [1, 4]

$$h(x,t) = \frac{(3v_4 - \xi)^2}{9g}, \quad v(x,t) = v_4 + \frac{2}{3}\,\xi, \quad 0 \le \xi = \frac{x}{t} \le v_2 - c_2, \quad c_2 = \sqrt{gh_2}. \tag{2.14}$$

The propagation speeds of the discontinuous waves are determined, as before, from formulas (2.10) and (2.11). The resulting solution, whose profile is shown by curve 2 in Fig. 2, will be called a solution of type B.

Solutions of type B are meaningful only if $D_3 < 0$ (D_3 is the speed of the reflected discontinuous wave). From this inequality it follows that on the adiabat u_s, the point (h_1, v_1) should lie below the point $G = (h_G, v_G)$, whose coordinates $h_G = h_1^*$ and $v_G = v_1^*$ are found from the equality $D_3 = 0$, which leads to the following system of equations:

$$\begin{gathered} v_1^* = v_s(h_1^*, z_0) = D(h_3^*, h_1^*), \qquad v_3^* = v_r(h_3^*, h_1^*, v_1^*), \qquad v_4^* = \sqrt{gh_4^*}, \\ J(h_3^*, q^*) = J(h_4^*, q^*) + \delta, \qquad q^* = h_1^* v_1^* = h_4^* v_4^*. \end{gathered} \tag{2.15}$$

For $D_3 = 0$, the reflected discontinuous wave coincides with the stationary discontinuity above the step, forming together with it a unified standing jump, at which there is a loss of the total energy of the incident flow. Since the flow ahead of the front of the standing jump is supercritical, the point G is on the part of the adiabat u_s located in the region of supercritical flows in Fig. 3.

Thus, self-similar solutions of the discontinuity decay problem (1.5)–(1.8) are constructed under the condition that the depth h_1 behind the front of the discontinuous wave incident on the step satisfies the inequalities $z_0 < h_1 < h_G$.

3. Self-Similar Solutions with Supercritical Flow ahead of the Step. We will construct self-similar solutions of the discontinuity decay problem (1.5)–(1.8) in which the constant supercritical flow (parameters h_1 and v_1) coincident with the flow behind the front of the initial discontinuous wave incident on the step (Fig. 4) is conserved to the left of the step. In such solutions, the total momentum of the flow incident on the step is sufficient to prevent upstream propagation of the effect of the step to the region $x < 0$ after the passage of the initial discontinuous wave over the step. The indicated self-similar solutions can be of two types: solutions of type C, in which conditions (2.4) representing the conservation of the total energy of the flow at the discontinuity above the step are satisfied, and solutions of type D, in which the total energy of the flow is lost at the step.

We first consider solutions of type C. In [11], it is shown that the mapping F_- specified by relations (2.4) transforms the critical-flow function (2.6) to a monotonically increasing function $v_c^- = F_-[v_c]$, whose plot is in the region of supercritical flows (Fig. 5). We denote by $P = (h_P, v_P)$ the point of intersection of the adiabat u_s with the curve v_c^-. The coordinates of this point are found from the system

$$v_P = v_s(h_P, z_0), \qquad J(h_P, q) = J(\sqrt[3]{q^2/g}, q) + \delta, \qquad q = h_P v_P.$$

In Fig. 5, the curve $u_s^- = F_-^{-1}[u_s]$, where F_-^{-1} is the mapping inverse of F_-, shows the image of the part of the adiabat u_s located above the point P in the region of supercritical flows. As shown in [11], in the neighborhood of the critical-flow curve, the function $u_s^-(h)$ is two-valued.

We assume that the flow behind the front of the discontinuous wave incident on the step is supercritical and the parameters of this flow h_1 and v_1 are not below the point P on the adiabat u_s, i.e., $h_1 \ge h_P$. Then, solving the discontinuity decay problem (1.5)–(1.8), we find that the constant flow (h_1 and v_1) persists to the left of the step; in view of the stability conditions obtained in [17], this flow forms a flow (h_4 and v_4) at the step which is critical at $h_1 = h_P$ and supercritical at $h_1 > h_P$. The parameters of this flow h_4 and v_4 satisfy the inequalities $h_4 > h_1$ and $v_4 < v_1$ and are determined as the image of the point (h_1, v_1) under the mapping F_-^{-1} or, what is the same, as the coordinates of the point of intersection of the curve u_s^- with the hyperbola $v_q(h, h_1, v_1)$.

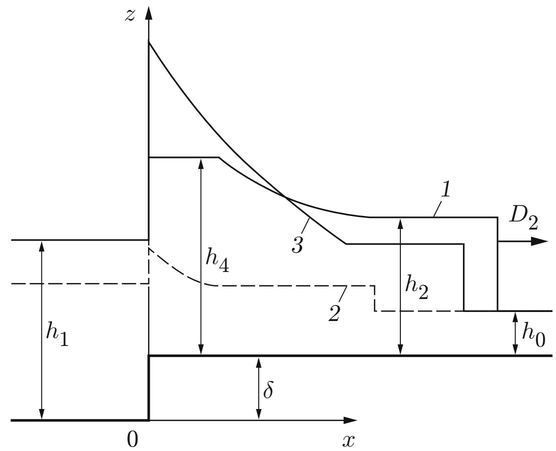

Fig. 4. Wave profiles that arise after the passage of the initial discontinuous wave over the bottom step in the case of supercritical flow ahead of the step: 1) flow described by a solution of type C; 2) limiting case of flow of type C; 3) flow described by a solution of type D.

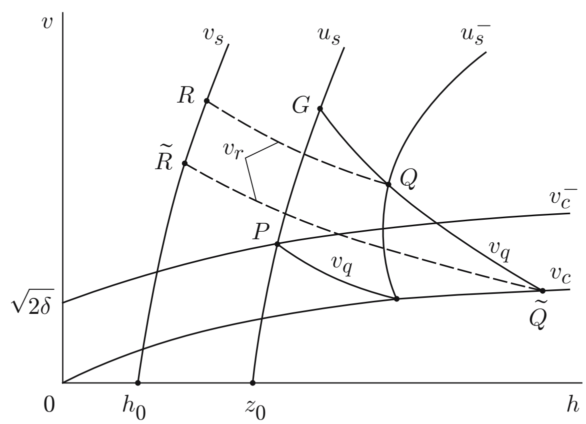

Fig. 5. Diagram of adiabats for constructing solutions of type C and D.

The solution to the right of the step is constructed, as for flows of type B, by soling the discontinuity decay problem (2.12) above an even bottom. The solution of this problem yields a discontinuous s-wave propagating at the speed (2.10) and a centered depression r-wave (2.14), in which $\xi \in [v_4 - c_4, v_2 - c_2]$. The constant-flow parameters h_2 and v_2 between these waves are calculated as the coordinates of the point of intersection of the shock s-adiabat (2.5) with the wave r-adiabat (2.13). The profile of the resulting solution of type C for $h_1 > h_P$ is shown by solid curve 1 in Fig. 4. The limiting case $h_1 = h_P$, where the left boundary of the depression wave is at the step is given by dashed curve 2 in Fig. 4.

We now consider solutions of type D. As shown in [8], there are two classes of solutions in which the total energy is lost at the bottom step: solutions in which two characteristics of system (1.1), (1.2) arrive at the discontinuity (1.6) and solutions in which three characteristics of this system arrive at the discontinuity (1.6). In the case of solutions of the first class, to close the shallow water model, one needs to modify the condition $[v^2/2+gz] = 0$ by introducing in it an heuristic parameter that specifies the part of the total energy of the flow that is lost in passing over through the bottom step. In the case of solutions of the second class, the continuity of the flow rate is sufficient for the closure of the conditions at the discontinuity (1.6) $[q] = 0$. In this case, the part of the total energy of the flow that is lost in passing over the step is uniquely determined within the framework of the shallow water model without invoking any heuristic parameters. In the present paper, we consider only the second class of solutions.

In order that in a solution of type D three characteristics arrive at the discontinuity above the step, it is necessary that the flow ahead of the step be supercritical and the flow at the step be critical. As shown in [8, 9], for the energy stability of such a discontinuity, which is related to the loss of the total energy at it, it is necessary that the flow parameters h_1 and v_1 lie above the point P on the adiabat u_s, i.e., it is necessary that the initial depth satisfy the inequality $h_1 > h_P$. If this inequality is satisfied, the discontinuity decay problem (1.5)–(1.8) admits energetically stable solutions of the second class in which the initial flow (h_1, v_1) is conserved to the left of the step, and a critical flow forms at the step, whose parameters $h_4 = \sqrt[3]{q^2/g}$ and $v_4 = \sqrt[3]{gq}$, where $q = h_1 v_1$, are calculated as the coordinates of the point of intersection of the hyperbola $v = v_q(h, h_1, v_1) = h_1 v_1/h$ with the critical-flow curve v_c. After determination of the values of h_4 and v_4, the flow to the right of the step is found, as in solutions of type B, by solving the discontinuity decay problem (2.12) above an even bottom. The profile of the resulting solution of type D is shown by curve 3 in Fig. 4. We note that in solutions of type C and D, as well as in solutions of type B, the flow to the right of the step is supercritical and does not influence the flow parameters at the step and to the left of it.

Since in the limiting case of solutions of type B, where $D_2 = 0 \Rightarrow h_1 = h_G$, these solutions continuously become energetically stable solutions of type D, the point G lies above the point P on the adiabat u_s (see Fig. 5). This implies that $h_G > h_P$, and hence at the initial depth $h_1 \in (h_P, h_G)$, where the parameters of the initial flow h_1 and v_1 lie between the points P and G on the adiabat u_s, the discontinuity decay problem (1.5)–(1.8) admits three different solutions: solutions of type B, C, and D. If $h_1 \geq h_G$, this problem admits two different solutions: solution of type C and solution of type D. Thus, for $h_1 \in (h_P, h_G)$ triple hysteresis occurs, and for $h_1 \geq h_G$, double hysteresis occurs.

As an example, curves GQR and $G\tilde{Q}\tilde{R}$ in Fig. 5 show a diagram of constructing solutions of type C and D obtained for the same initial depth $h_1 = h_G$ at which there is a continuous transition of solutions of type B to solutions of type D. The coordinates of the points $Q = (h_4, v_4)$, $\tilde{Q} = (\tilde{h}_4, \tilde{v}_4)$, $R = (h_2, v_2)$, and $\tilde{R} = (\tilde{h}_2, \tilde{v}_2)$, which specify the parameters of the indicated solutions, satisfy the inequalities

$$h_4 < \tilde{h}_4, \quad v_4 > \tilde{v}_4, \quad h_2 > \tilde{h}_2, \quad v_2 > \tilde{v}_2 \quad \Rightarrow \quad D_2 > \tilde{D}_2,$$

according to which curves 1 and 3 are plotted in Fig. 4.

As $h_1 \to h_P + 0$, i.e., as the parameters of the initial flow h_1 and v_1 tend from above along the adiabat u_s to the point P (see Fig. 5), solutions of type D continuously become the limiting solution of type C, which is shown by the dashed curve 2 in Fig. 4. However, this limiting solution is unstable against small changes in the initial data: for $h_1 = h_P - \varepsilon$ ($\varepsilon \ll 1$), it suddenly becomes a solution of type B with a reflected discontinuous wave and subcritical flow (h_3, v_3) ahead of the step (see curve 2 in Fig. 2).

4. Regions of Existence of Constructed Solutions on the Plane of Determining Parameters. Performing a homothetic transformation with respect to the time and space variables, we transform to dimensionless values, for which $g = z_0 = 1$. In this case, the solutions of the generalized problem of discontinuity decay (1.5)–(1.8) are completely determined by the following two dimensionless parameters: the relative height of the step $\delta = \delta/z_0 \in (0, 1)$ and the relative level (depth) of water $z_1 = h_1 = z_1/z_0 > 1$ behind the front of the initial discontinuous wave incident on the step. Since the dimensionless speed $v_1 = v_1/\sqrt{gz_0}$, which can be treated as the Froude number in the problem in question, is uniquely related to the initial level z_1 by the shock adiabat equation $v_1 = u_s(z_1) = v_s(z_1, z_0)$, the introduced pair of determining parameters δ and z_1 is similar to the pair of parameters δ and v_1, which is often used in the analysis of problems of single-layer flow over a bottom obstacle [3, 7, 19]. In our case, the parameters δ and z_1 are more convenient than parameters δ and v_1 since the direct relation $v_1 = v_s(z_1, z_0)$ on the shock adiabat (1.8) is simpler and more generally accepted than its inverse relation $z_1 = v_s^{-1}(v_1, z_0)$. In view of this, the regions of existence of self-similar solutions of type A, B, C, and D in Fig. 6 are given on the plane of the dimensionless determining parameters δ and z_1.

The region of existence of solutions of type A (a typical profile of these solutions is shown by curve 1 in Fig. 2) is located in Fig. 6 below curve 1, whose equation in the form of the dependence $z_1(\delta)$ is determined as follows. From the relations $v_A = v_s(h_A, h_0) = \sqrt{h_A}$, where $h_0 = 1 - \delta$, we first calculate the coordinates $h_A(\delta)$ and $v_A(\delta)$ of the point A at which the adiabat v_s is intersected with the critical-flow curve v_c in Fig. 3. As shown in [20],

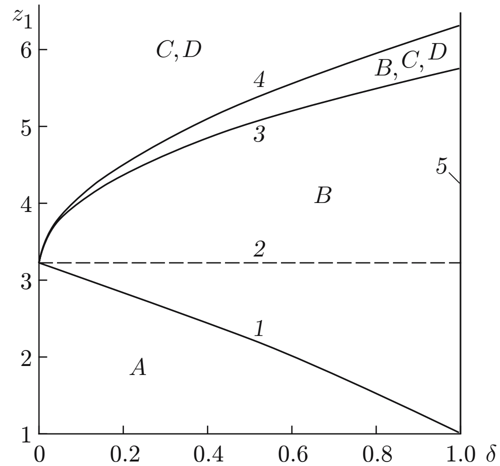

Fig. 6. Regions of existence of solutions on the plane of the dimensionless determining parameters: 1) the curve separating the regions of existence of solutions of type A and B; 2) the critical-flow curve; 3) the lower boundary of the region of existence of solutions of type C and D; 4) the upper boundary of the region of existence of solutions of type B; 5) the right boundary of the regions of existence of the solutions.

$$h_A(\delta) = z^*(1-\delta) \quad \Rightarrow \quad v_A(\delta) = \sqrt{h_A(\delta)} = \sqrt{z^*(1-\delta)},$$

where

$$z^* = 1 + \frac{4}{\sqrt{3}} \cos\Big(\frac{1}{3} \arccos \frac{3\sqrt{3}}{8}\Big) \approx 3.214 \tag{4.1}$$

is the maximum root of the cubic equation $z^3 - 3z^2 - z + 1 = 0$. After that, from the system

$$J(h_B, q) = J(h_A(\delta), q) + \delta, \qquad q = h_A(\delta) v_A(\delta) = h_B v_B \tag{4.2}$$

we find the coordinates $h_B(\delta)$ and $v_B(\delta)$ of the point B in Fig. 3, at which the curves v_s^+ and v_c^+ are intersected. Then, from the system

$$v_B(\delta) = v_r(h_B(\delta), h_C, v_C), \qquad v_C = v_s(h_C, 1) \tag{4.3}$$

we determine the coordinates $h_C(\delta)$ and $v_C(\delta)$ of the point C in Fig. 3 at which the shock adiabats u_s and v_r are intersected. As a result, the required relation $z_1(\delta)$ is specified by the function $z_1 = h_C(\delta)$, whose plot is constructed by numerical solution using the method of iterations of systems (4.2) and (4.3) for various values of $\delta \in (0, 1)$.

In Fig. 6, the region of existence of solutions of type B (whose typical profile is shown by curve 2 in Fig. 2) is located between curves 1 and 4. The equation of curve 4 in the form of an explicit dependence $\delta(z_1)$, where $z_1 > z^*$, is obtained from system (2.15) and can be written as

$$\delta = J(h_3, q) - J(h_4, q) = (v_3^2 - v_4^2)/2 + h_3 - h_4,$$

where

$$v_3 = q/h_3, \quad h_3 = z_1(\sqrt{1 + 8f_1^2} - 1)/2, \quad f_1 = v_1/\sqrt{z_1}$$

are the relations at the front of the reflected discontinuous wave for the case where it becomes a standing hydraulic jump for $D_3 = 0$;

$$v_4 = \sqrt[3]{q}, \qquad h_4 = \sqrt[3]{q^2}, \qquad q = z_1 v_1 \tag{4.4}$$

are the critical-flow parameters at the step obtained as the coordinates of the point $\tilde{Q}$ in Fig. 5 at which the hyperbola $v_q = q/h$ issuing from the point G is intersected with the critical-flow curve v_c;

$$v_1 = v_s(z_1, 1) = (z_1 - 1)\sqrt{(z_1 + 1)/(2z_1)} \tag{4.5}$$

is the relation (1.7) on the shock adiabat (1.8) obtained for $z_0 = 1$.

Since solutions of type C and D (typical profiles of these solutions are shown by curves 1 and 3 in Fig. 4) occur when the point (z_1, v_1) lies on the shock adiabat u_s above the point P at which these adiabats are intersected with the curve $v_c^- = F_-[v_c]$ in Fig. 5, the region of existence of solutions of type C and D lies above curve 3 in Fig. 6. The equation of curve 3 is obtained by writing the function $v_c^-(h)$ as the following explicit dependence $\delta(z_1)$:

$$\delta = J(z_1, q) - J(h_4, q) = (v_1^2 - v_4^2)/2 + z_1 - h_4.$$

Here the quantities v_4, h_4, and v_1, as functions of $z_1 > z^*$, are specified by formulas (4.4) and (4.5). The dashed curve 2 in Fig. 6 shows the critical level $z_1 = z^* \approx 3.214$, at which the flow behind the front of the discontinuous wave incident on the step is critical. For such an initial level of z_1, flow of type B always occurs.

From the above analysis it follows that if the parameters δ and z_1 lie below curve 3 in the diagram given in Fig. 6, then the solution of the generalized discontinuity decay problem (1.5)–(1.8) is determined uniquely. If these parameters are above curve 3, then hysteresis takes place. Between curves 3 and 4, this hysteresis is triple (solutions of type B, C, and D can exist simultaneously), and above curve 4, it is double (solutions of type C and D can exist simultaneously).

5. Analogy with the Problem of Flow past a Bottom Obstacle. A comparison of the diagram given in Fig. 6 with a similar diagram of the regions of existence of the various solutions in the problem of single-layer fluid flow past a bottom obstacle [3, 17–19] shows that this problem is in many respects similar to the problem of the incidence of a discontinuous wave on a bottom step considered in the present study. In diagram of [17] also has four main regions similar to those shown in Fig. 6. In this case, the solution of type A and the solution of type C correspond to completely subcritical and completely supercritical flows over a bottom obstacle, respectively, with only local perturbation of the incident flow. The solution of type B corresponds to flow at which the obstacle blocks [17] (controls [3]) the upstream current, forming a reflected discontinuous wave. At the top of the obstacle there is a transition from subcritical to supercritical flow.

In the problem of flow over a bottom obstacle there is a hysteresis region [3, 17] similar to that located between curves 3 and 4 in Fig. 6. In this region both the completely supercritical and blocking solutions of this problem can exist simultaneously (this result was supported experimentally in [18]). In [19], it is shown that in the same region of determining parameters there is a third solution similar to the solution of type D, in which a standing hydraulic jump forms at the upstream slope of the obstacles. In a linear approximation, this hydraulic jump is unstable [19], but this does not imply that it is instable within the framework of the general nonlinear problem in the case of small flow variations ahead of the obstacle and at its top. Indirect evidence of this is provided by the results of a study [21], in which a steady-state standing hydraulic jump ahead of the bottom step was implemented experimentally in a laboratory tank.

From [19] it follows that if the bottom step (1.6) models a monotonic elevation of a shelf-type bottom [9], whose length far exceeds the width of the hydraulic jump located on it, the solution of type D describing this flow ignoring bottom friction can exist only in the region of triple hysteresis, which corresponds to the segment of the adiabat u_s between the points P and G in Fig. 5 and the region between curves 3 and 4 in Fig. 6. However, if the bottom step (1.6) models a discontinuity of the bottom level of a real channel, whose roughness factor ahead of the step is large enough, the energy loss at the standing jump ahead of the step can be much higher than that in a similar standing jump above a smooth horizontal bottom. In this case, the solution of type D describing such flow corresponds to the segment of the adiabat u_s lying above the point G in Fig. 5 and the region above curve 4 in Fig. 6. Thus, at for initial depth $h_1 > h_G$, double hysteresis can occur.

6. Proof of Theorem 1. We denote by (h_A, v_A) and (h_F, v_F) the coordinates of the points A and F, at which the adiabats v_s and u_s intersect the critical-flow curve v_c in Fig. 3. Because the adiabat u_s for $h > h_F$ lies in the region of supercritical flows and the plot of the function v^+ defined by formula (2.7) is in the region of subcritical flows for all $h > z_0$, it follows that to prove Theorem 1, it suffices to show that inequality (2.8) is valid for all $h \in (z_0, h_F)$.

Let us consider the function

$$v = v(h) = \begin{cases} v_s(h), & h_0 < h \le h_A, \\ v_c(h), & h \ge h_A, \end{cases} \tag{6.1}$$

whose image for the mapping F_+ is the function v_+. The plot of the function (6.1) consists of the part of the adiabat v_s located in the region of subcritical flows and the part of the critical-flow curve v_c located to the right of the point A in Fig. 3. Let $A_1 = (\hat{h}, \hat{v})$ be a certain point that lies on the plot of the function (6.1) and satisfies the condition $h_0 < \hat{h} < h_F$. We draw the hyperbola $v = v_q(h, \hat{h}, \hat{v})$ through the point A_1. At some points $B_1 = (h_+, v_+)$ and $C_1 = (H, V)$, this hyperbola intersects the curve v_+ and the adiabat u_s. To prove inequality (2.8), it suffices to show that the point C_1 lies on the hyperbola v_q to the left of the point B_1. The latter, as follows from [8], is equivalent to satisfying the following inequality for $H > \hat{h}$:

$$J(\hat{h}, q) + \delta > J(H, q), \qquad q = \hat{h}\hat{v} = HV. \tag{6.2}$$

This inequality implies that in transition from the flow ahead of the step (with parameters H and V) to the flow at the step (with parameters $\hat{h}$ and $\hat{v}$), the total energy of the incident flow increases.

We assume that the quantities h_0, $\hat{h}$, and $\hat{v}$ and, hence, $q = \hat{h}\hat{v}$, are fixed. Then, in view of the relation $\delta = z_0 - h_0$, inequality (6.2) can be written as

$$\varphi(H, z_0) < \varphi(\hat{h}, h_0) \quad \forall H > \hat{h}, \qquad \varphi(h, z) = q^2/(2gh^2) + h - z, \tag{6.3}$$

where the quantities H and z_0 are related by the condition

$$HV = Hu_s(H) = H(H - z_0)\sqrt{g(H + z_0)/(2Hz_0)} = q. \tag{6.4}$$

From condition (6.4), the initial level z_0 can be expressed as a function $z_0 = z(H)$ such that $z(\hat{h}) = h_0$ for $\hat{h} \le h_A$ and $z(\hat{h}) > h_0$ for $\hat{h} > h_A$. This implies that inequality (6.3) is satisfied provided that

$$\varphi(H, z(H)) < \varphi(\hat{h}, z(\hat{h})) \quad \forall H > \hat{h}. \tag{6.5}$$

To prove inequality (6.5), it suffices to show that

$$\frac{d}{dh}\,\varphi(h, z(h)) = 1 - \frac{q^2}{gh^3} - z_h < 0 \quad \forall h > z(h). \tag{6.6}$$

We introduce an auxiliary symmetric function $\psi(h, z) = (h - z)^2(h + z)$. Using this function, condition (6.4), to which the quantities h and $z(h)$ satisfy, can be written as

$$h\psi(h, z(h))/z(h) = 2q^2/g = \text{const}. \tag{6.7}$$

Differentiating relation (6.7) with respect to h and taking into account that

$$\psi_h = (h - z)(3h + z), \qquad \psi_z = (z - h)(3z + h),$$

we obtain

$$z_h = \frac{z(\psi + h\psi_h)}{h(\psi - z\psi_z)} = \frac{z(4h^2 + zh - z^2)}{h(h^2 + zh + 2z^2)} = \frac{4y^2 + y - 1}{y(y^2 + y + 2)}, \tag{6.8}$$

where $y = h/z > 1$.

In view of (6.8), we have

$$1 - z_h = \frac{y^3 - 4y^2 + 2y + 1}{y(y^2 + y + 2)} = \frac{(y - 1)(y^2 - 3y - 1)}{y(y^2 + y + 2)},$$

and in view of (6.4),

$$\frac{q^2}{gh^3} = \frac{u_s^2(h)}{gh} = \frac{(h + z)(h - z)^2}{2zh^2} = \frac{(y + 1)(y - 1)^2}{2y^2}.$$

Therefore, inequality (6.6) is written as

$$\frac{(y + 1)(y - 1)^2}{2y} > \frac{(y - 1)(y^2 - 3y - 1)}{y^2 + y + 2}. \tag{6.9}$$

For $y > 1$, the validity of inequality (6.9) follows from the formulas

$$(y^2 - 1)(y^2 + y + 2) - 2y(y^2 - 3y - 1) = y^4 - y^3 + 7y^2 + y - 2 = (y - 1)(y^3 + 2y + 2) + 5y^2 + y > 0.$$

Theorem 1 is proved.

7. Proof of Theorem 2. To prove Theorem 2, it suffices to show that the point C lies on the adiabat u_s (see Fig. 3) below the point F, which is equivalent to the inequality $h_C < h_F$. By virtue of Theorem 1, the adiabat u_s is above the curve v_c^+ in Fig. 3 and the adiabat v_r, which passes through the points B and C, is strictly monotonically decreasing, $h_C < h_B$. This implies that the inequality $h_C < h_F$ follows from the inequality $h_B < h_F$. To prove the latter, we note that in view of (2.4), the depth h_B, which will be denoted by H, is calculated by the equation $J(H, q) = J(h_A, q) + \delta$, where $q^2 = h_A^2 v_A^2 = g h_A^3$. The expansion of this equation has the form

$$h_A^3/(2H^2) + H - z = 3h_A/2 - h_0, \tag{7.1}$$

where $z = z_0$ is the initial level ahead of the step.

Let us fix the depths h_0 and h_A ahead of the step and consider the depths $H = h_B$ and h_F as functions of the initial level $z > h_0$. Since

$$\lim_{z \to h_0} H(z) = \lim_{z \to h_0} h_F(z) = h_A,$$

to prove the inequality $H = h_B < h_F$ for $z > h_0$ it suffices to show that

$$H'_z < (h_F)'_z \quad \forall z > h_0. \tag{7.2}$$

To calculate the derivative H'_z, we differentiate Eq. (7.1) with respect to z. As a result, taking into account that $H > h_A$ for $z > h_0$, we obtain

$$H'_z = 1 + h_A^3/H^3 < 2 \quad \forall z > h_0. \tag{7.3}$$

As shown in [20], $h_F(z) = z^* z$, where z^* is a constant defined by formula (4.1). This implies that

$$(h_F)'_z = z^* > 3. \tag{7.4}$$

A comparison of inequalities (7.3) and (7.4) yields the unknown inequality (7.2). Theorem 2 is proved.

Conclusions. The analysis of the problem of the flows resulting from the incidence of a discontinuous wave on a bottom step showed that this problem is always solvable within the framework of the self-similar solutions of shallow water theory but it is not uniquely solvable. If the constant flow (h_1, v_1) behind the front of the initial discontinuous wave is supercritical and $h_1 > h_P$ (see Fig. 5), i.e., if the parameters of the problem (1.5)–(1.8) are above curves 3 in Fig. 6, this problem admits two different solutions with supercritical flow ahead of the step: a solution of type C, in which the total energy is conserved at the step (this solution is shown by curve 1 in Fig. 4), and a solution of type D, in which the total energy is lost at the step (this solution is given by curve 3 in Fig. 4). For the stronger limitation $h_P < h_1 < h_G$, i.e., if the parameters of the problem (1.5)–(1.8) lie between curves 3 and 4 in Fig. 6, this problem also admits a solution of type B with a reflected discontinuous wave and subcritical flow ahead of the step. In this solution, which is shown by curve 2 in Fig. 2, the total energy is conserved at the step.

The many-valuedness of the solutions of the problem (1.5)–(1.8) is due to the fact that the shallow water equations are the long-wave approximation of the Euler equations [1, 4], by virtue of which the discontinuous shallow water solutions can describe the entire transition regions of rapid changes in real-flow parameters. In particular, discontinuous solutions above a bottom-level jump can be used to model wave flows in the cases where the bottom level of a real channel undergoes a discontinuity or when it has a segment of sharp monotonic change. For example, in [12], a bottom step in theory modeled a bottom level discontinuity in experiment, and in [9] a step in theory modeled a sharp elevation of a shelf-type bottom in experiment. In both cases in the solution of the dam break problem, theory is in good agreement with experiment for the possible types of waves, their propagation speeds, and the asymptotic depths behind their fronts.

Using this approach to the problem considered in the present paper, it can be assumed that if the bottom step (1.6) in theory corresponds to a bottom level discontinuity in experiment that offers the maximum possible resistance to the incident flow, then in laboratory simulations of the problem (1.5)–(1.8) with the initial depth $h_1 \in (h_P, h_G)$ (i.e, when the parameters of the problem are in the region of triple hysteresis located between curves 3 and 4 in Fig. 6), one should expect the occurrence of wave flow of type B with a reflected discontinuous wave. If, as in [9], the bottom step (1.6) models a monotonic elevation of a shelf-type bottom, then in an experiment in which a discontinuous wave of depth $h_1 \in (h_P, h_G)$ behind its front is incident on this shelf, one might expect the formation of flows of type C or D with supercritical flow ahead of the shelf. Thus in flows of type D, a standing hydraulic jump forms on the shelf, at which part of the total energy of the incident flow becomes the energy of vortical mixing, which is described in shallow water theory as a loss of the total energy. In flows of type C, supercritical flow forms on the shelf, for which the total energy is conserved at the bottom step (1.6) in shallow water. In the case of the same initial depth $h_1 > h_P$, transition from flows of type D to flows of type C occurs when the shelf steepness decreases. A more precise answer to the question of the conditions for the occurrence of flows of type B, C, and D can be obtained from numerical simulations of the problem of the incidence of a discontinuous wave on a shelf and from laboratory experiments using large tanks to produce a large initial level difference.

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